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BRESSOUD'S GENERALIZATION OF SCHUR'S THEOREM EXTENSION TO OVERPARTITIONS

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Abstract. Recently Jeremy Lovejoy [A Theorem on seven-colored overpartitions and its Applications], [5], has proved a q-series identity in four parameters and deduced an overpartition analogue of a classical partition theorem of Schur. In this paper we extend Bressoud's generalization of Schur's theorem for overpartitions.

1. Introduction

An overpartition is a partition in which the first occurrence of a number may be overlined. Recently Jeremy Lovejoy, S. Corteel and A. Yee [2] have studied overpartitions and obtained many interesting results. Lovejoy has extended Gordon's theorem [3] and has got Rogers–Ramanujan type theorems [4] for overpartitions. Further Jeremy Lovejoy [5] obtains the following overpartition analogue of the well-known partition theorem of Schur.

Theorem 1. *Let $A_k(n)$ denote the number of overpartitions of n into parts $\equiv 1$ or $2 \pmod 3$ with k non-overlined parts. Let $B_k(n)$ denote the number of overpartitions of n where parts differ by at least 3 if the smaller is overlined or both parts are divisible by 3, parts differ by at least 6 if the smaller is overlined and both parts are divisible by 3 AND there are k non-overlined parts. Then,*

$$A_k(n) = B_k(n) \qquad \text{for all } k \text{ and } n.$$

Padmavathamma et al [6] gave a simple combinatorial Proof of Theorem 1. In 1980, D.M. Bressoud [1] gave a combinatorial proof of Schur's 1926 theorem by establishing a one-to-one correspondence between the two types of partitions counted in the theorem. In fact he proves the following generalized version of Schur's theorem.

Theorem 2 (Generalized version of Schur's theorem)**.** *Given positive integers r and m such that $r < \frac{m}{2}$, let $C_{r,m}(n)$ denote the number of partitions of n into distinct parts $\equiv \pm\, r \pmod m$ and let $D_{r,m}(n)$ denote the number of partitions of n into distinct parts $\equiv 0, \pm\, r \pmod m$ with minimal difference m, minimal difference $2m$ between multiples of m. Then*

$$C_{r,m}(n) \;=\; D_{r,m}(n) \quad \text{for all } n.$$

Recently Padmavathamma et al [7] gave a simple bijective Proof of Theorem 2. The object of this paper is to extend Theorem 2 for overpartitions.

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Theorem 3 (Extension for overpartitions)**.** *Given positive integers k, r and m such that $r < \frac{m}{2}$, let $C_{k,r,m}(n)$ denote the number of overpartitions of n into parts $\equiv \pm\, r \pmod{m}$ with k non-overlined parts. Let $D_{k,r,m}(n)$ denote the number of partitions of n into parts $\equiv 0, \pm\, r \pmod{m}$ where parts differ by at least m if the smaller is overlined or both parts are divisible by m, parts differ by at least $2m$ if the smaller is overlined and both parts are divisible by m AND there are k non-overlined parts. Then*

$$C_{k,r,m}(n) \;=\; D_{k,r,m}(n) \quad \textit{for all} \;\; n.$$

Proof. We construct a mapping from the partitions enumerated by $C_{k,r,m}(n)$ to those enumerated by $D_{k,r,m}(n)$. Let $\pi = b_1 + b_2 + \cdots + b_s$ denote a partition enumerated by $C_{k,r,m}(n)$. If there is no pair $(b_i, \overline{b_{i+1}})$ or $(\overline{b_i}, \overline{b_{i+1}})$ in π such that $b_i - b_{i+1} \;<\; m$ then π is a partition enumerated by $D_{k,r,m}$ also since all the parts in π are $\equiv \pm r \pmod{m}$. We adopt the following procedure to go from $C_{k,r,m}(n)$ to $D_{k,r,m}(n)$. □

Step $\boldsymbol{CD_1}$. Arrange the parts of π in a column in decreasing order.

Step $\boldsymbol{CD_2}$. From the **top** look for the first i say i_1 for which $b_{i_1} - b_{i_1+1} < m$ and b_{i_1+1} is overlined. Here we get two possibilities.

$$\left(\begin{array}{c} b_{i_1} \\ \hline b_{i_1+1} \end{array}\right) \qquad \text{and} \qquad \left(\begin{array}{c} \overline{b_{i_1}} \\ \hline b_{i_1+1} \end{array}\right).$$

We replace the pair

$$\left(\begin{array}{c} b_{i_1} \\ \hline b_{i_1+1} \end{array}\right) \text{ by } (b_{i_1} + b_{i_1+1}) \qquad \text{and} \qquad \left(\begin{array}{c} \overline{b_{i_1}} \\ \hline b_{i_1+1} \end{array}\right) \text{ by } (\overline{b_{i_1} + b_{i_1+1}}).$$

We note that $b_{i_1} + b_{i_1+1} \;\equiv 0 \pmod{m}$ for the following reason:
Since all the parts in π are $\equiv \pm r \pmod{m}$ and $b_{i_1} - b_{i_1+1} < m$ only the following two possibilities can occur.

(i) $b_{i_1} = m(t+1) - r$ and $b_{i_1+1} = mt + r$

or

(ii) $b_{i_1} = mt + r$ and $b_{i_1+1} = mt - r$.

In the first case $(b_{i_1} + b_{i_1+1}) = m(2t+1)$ while in the second case $(b_{i_1} + b_{i_1+1}) = m(2t)$.

Eg: Let $k = 2$, $m = 10$ and $r = 3$

$$\text{(i)} \;\; \begin{array}{c} 13 \\ 7 \\ \overline{3} \end{array} \longrightarrow \begin{array}{c} 13 \\ 10 \end{array} \qquad\qquad \text{(ii)} \;\; \begin{array}{c} 33 \\ \overline{13} \\ \overline{7} \\ 3 \end{array} \longrightarrow \begin{array}{c} 33 \\ \overline{20} \\ 3 \end{array}$$

Step $\boldsymbol{CD_3}$. The replacement of $(b_{i_1} + b_{i_1+1})$ or $(\overline{b_{i_1} + b_{i_1+1}})$ gives rise to the following possibilities.

<u>CASE 1:</u>

$$\begin{pmatrix} b_{i_1-1} \\ b_{i_1}+b_{i_1+1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \overline{b_{i_1-1}} \\ b_{i_1}+b_{i_1+1} \end{pmatrix}$$

where $b_{i_1}+b_{i_1+1} \ < \ b_{i_1-1}$ in which case we proceed to the next step.

Eg: Let $k=2$, $m=4$ and $r=1$

$$\text{(i)} \ \begin{array}{c} 5 \\ 3 \\ \overline{1} \end{array} \longrightarrow \begin{array}{c} 5 \\ 4 \end{array} \qquad\qquad \text{(ii)} \ \begin{array}{c} 17 \\ \overline{9} \\ 5 \\ \overline{3} \end{array} \longrightarrow \begin{array}{c} 17 \\ \overline{9} \\ 8 \end{array}$$

<u>CASE 2:</u>

$$\begin{pmatrix} b_{i_1-1} \\ b_{i_1}+b_{i_1+1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \overline{b_{i_1-1}} \\ b_{i_1}+b_{i_1+1} \end{pmatrix}$$

where $b_{i_1}+b_{i_1+1} \ > \ b_{i_1-1}$. In this case we replace the first one by $\begin{pmatrix} b_{i_1}+b_{i_1+1} \\ b_{i_1-1} \end{pmatrix}$ while we replace the second one by $\begin{pmatrix} \overline{b_{i_1}+b_{i_1+1}} \\ b_{i_1-1} \end{pmatrix}$.

Eg: Let $k=2$, $m=5$ and $r=2$

$$\text{(i)} \ \begin{array}{c} 8 \\ 7 \\ \overline{3} \end{array} \longrightarrow \begin{array}{c} 8 \\ 10 \end{array} \longrightarrow \begin{array}{c} 10 \\ 8 \end{array} \qquad\qquad \text{(ii)} \ \begin{array}{c} 22 \\ \overline{13} \\ 8 \\ \overline{7} \end{array} \longrightarrow \begin{array}{c} 22 \\ \overline{13} \\ 15 \end{array} \longrightarrow \begin{array}{c} 22 \\ \overline{15} \\ 13 \end{array}$$

<u>CASE 3:</u>

$$\begin{pmatrix} b_{i_1-1} \\ \overline{b_{i_1}+b_{i_1+1}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \overline{b_{i_1-1}} \\ \overline{b_{i_1}+b_{i_1+1}} \end{pmatrix}$$

where $b_{i_1-1}-(b_{i_1}+b_{i_1+1}) \ > \ m$.

In this case we proceed to the next step.

Eg: Let $k=1$, $m=5$ and $r=1$

$$\text{(i)} \ \begin{array}{c} 11 \\ \overline{4} \\ \overline{1} \end{array} \longrightarrow \begin{array}{c} 11 \\ \overline{5} \end{array} \qquad\qquad \text{(ii)} \ \begin{array}{c} \overline{16} \\ \overline{6} \\ \overline{4} \\ 1 \end{array} \longrightarrow \begin{array}{c} \overline{16} \\ \overline{10} \\ 1 \end{array}$$

Case 4:

$$\begin{pmatrix} b_{i_1-1} \\ \overline{b_{i_1}+b_{i_1+1}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \overline{b_{i_1-1}} \\ \overline{b_{i_1}+b_{i_1+1}} \end{pmatrix}$$

where $b_{i_1-1} - (b_{i_1} + b_{i_1+1}) < m$. In this case we replace the first one by $\begin{pmatrix} b_{i_1}+b_{i_1+1}+m \\ \overline{b_{i_1-1}-m} \end{pmatrix}$ while we replace the second one by $\begin{pmatrix} \overline{b_{i_1}+b_{i_1+1}+m} \\ \overline{b_{i_1-1}-m} \end{pmatrix}$.

Eg: Let $k = 1$, $m = 7$ and $r = 2$

(i) $\begin{array}{c} 12 \\ \overline{5} \\ \overline{2} \end{array} \longrightarrow \begin{array}{c} 12 \\ \overline{7} \end{array} \longrightarrow \begin{array}{c} 14 \\ \overline{5} \end{array}$ (ii) $\begin{array}{c} \overline{16} \\ \overline{9} \\ \overline{5} \\ 2 \end{array} \longrightarrow \begin{array}{c} \overline{16} \\ \overline{14} \\ 2 \end{array} \longrightarrow \begin{array}{c} \overline{21} \\ \overline{9} \\ 2 \end{array}$

Note. In the above cases $b_{i_1} + b_{i_1+1} \neq b_{i_1-1}$ and $b_{i_1-1} - (b_{i_1} + b_{i_1+1}) \neq m$ because $b_{i_1-1} \not\equiv 0 \pmod m$ and $b_{i_1} + b_{i_1+1} \equiv 0 \pmod m$.

We apply Step CD_3 repeatedly till difference condition is satisfied for the parts of π from the top upto the i_1^{th} position.

Eg: Let $k = 3$, $m = 6$ and $r = 2$

$$\begin{array}{c} 20 \\ 16 \\ \left.\begin{array}{c} \overline{10} \\ \overline{8} \end{array}\right\} \\ 4 \\ \overline{2} \end{array} \longrightarrow \begin{array}{c} 20 \\ \left.\begin{array}{c} 16 \\ \overline{18} \end{array}\right\} \\ 4 \\ \overline{2} \end{array} \longrightarrow \begin{array}{c} \left.\begin{array}{c} 20 \\ 24 \end{array}\right\} \\ \overline{10} \\ 4 \\ \overline{2} \end{array} \longrightarrow \begin{array}{c} 24 \\ 20 \\ \overline{10} \\ 4 \\ \overline{2} \end{array}$$

Step CD_4. Look for the next i say i_2 for which difference condition is not satisfied. The same procedure explained in Step CD_2 and Step CD_3 is carried out till difference condition is satisfied for all the parts from the top upto the i_2^{th} position.

The difference condition between multiples of m is clearly satisfied in our mapping for the following reasons.

(i) If $\begin{pmatrix} a \\ \overline{b} \\ c \\ \overline{d} \end{pmatrix} \longrightarrow \begin{pmatrix} a+b \\ c+d \end{pmatrix}$ then $a + b \geq c + d + m$ even if $b = c$,

(ii) If $\begin{pmatrix} a \\ \overline{b} \\ \overline{c} \\ \overline{d} \end{pmatrix} \longrightarrow \begin{pmatrix} a+b \\ \overline{c+d} \end{pmatrix}$ then $a + b \geq c + d + 2m$ since $b \neq c$,

(iii) If $\begin{pmatrix} a \\ \bar{b} \\ c \\ \bar{d} \\ \bar{e} \end{pmatrix} \longrightarrow \begin{pmatrix} a+b \\ c \\ \overline{d+e} \end{pmatrix}$ then $a+b \geq d+e+3m$ even if $b = c$ since $c \geq d+m$

and neither of a, b, c, d, e is $\equiv 0 \pmod{m}$.

Proceeding like this (in a finite number of steps) we arrive at a stage where difference condition will be satisfied for all the parts of π. The resulting partition is the required partition enumerated by $D_{k,r,m}(n)$.

We illustrate our procedure by an example by taking $k = 4$, $m = 20$ and $r = 7$.

Let $$\pi = \overline{107} + 87 + \overline{73} + 53 + 53 + \overline{47} + \overline{33} + \overline{13} + 13 + \overline{7}$$

be a partition enumerated by $C_{4,7,20}(486)$.

$$C_{k,r,m}(n) \to D_{k,r,m}(n)$$

$$\begin{array}{c} \overline{107} \\ \left.\begin{array}{c} 87 \\ \overline{73} \end{array}\right\} \\ 53 \\ 53 \\ \overline{47} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \to \begin{array}{c} \left.\begin{array}{c} \overline{107} \\ 160 \end{array}\right\} \\ 53 \\ 53 \\ \overline{47} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ 53 \\ \left.\begin{array}{c} 53 \\ \overline{47} \end{array}\right\} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ \left.\begin{array}{c} 53 \\ 100 \end{array}\right\} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \overline{33} \\ \overline{13} \\ \left.\begin{array}{c} 13 \\ \overline{7} \end{array}\right\} \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \overline{33} \\ \left.\begin{array}{c} \overline{13} \\ 20 \end{array}\right\} \end{array} \to$$

$$\begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \left.\begin{array}{c} \overline{33} \\ \overline{20} \end{array}\right\} \\ 13 \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ \left.\begin{array}{c} 53 \\ \overline{40} \end{array}\right\} \\ \overline{13} \\ 13 \end{array} \to \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 60 \\ \overline{33} \\ \overline{13} \\ 13 \end{array}$$

The last partition is the associated partition of π enumerated by $D_{4,7,20}(486)$.

We now give the reverse mapping from $D_{k,r,m}(n)$ to $C_{k,r,m}(n)$. Let ψ be a partition enumerated by $D_{k,r,m}(n)$. If no part is a multiple of m, then it is a partition enumerated by $C_{k,r,m}(n)$ also. We adopt the following procedure to go from $D_{k,r,m}(n)$ to $C_{k,r,m}(n)$.

Step DC_1. Let the parts of ψ be arranged in a column in decreasing order.

Step DC_2. From the **bottom** look for the first multiple of m say x. If there is no part lying below x, then we split x into (α, β) as detailed below in the table.

$x = m * 2t$	$x \to (\alpha, \overline{\beta}) = (mt + r, \overline{mt - r})$ $\overline{x} \to (\overline{\alpha}, \overline{\beta}) = (\overline{mt + r}, \overline{mt - r})$	Eg : $m = 10$ and $r = 4$ $20 = 10 * (2.1) \to 14 + \overline{6}$ $\overline{40} = 10 * (2.2) \to \overline{24} + \overline{16}$
$x = m * (2t + 1)$	$x \to (\alpha, \overline{\beta}) = (m(t + 1) - r, \overline{mt + r})$ $\overline{x} \to (\overline{\alpha}, \overline{\beta}) = (\overline{m(t + 1) - r}, \overline{mt + r})$	Eg : $m = 10$ and $r = 4$ $30 = 10 * (2.1 + 1) \to 16 + \overline{14}$ $\overline{50} = 10 * (2.2 + 1) \to \overline{26} + \overline{24}$

Here we observe that parts α and β are $\equiv \pm r \pmod{m}$. Suppose there is a part $y \equiv \pm r \pmod{m}$ lying below x, then we get the following possibilities.

<u>Case 1:</u> If y is not overlined and $y < \alpha$, then we replace

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \alpha \\ \overline{\beta} \\ y \end{pmatrix} \text{ and } \begin{pmatrix} \overline{x} \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \overline{\alpha} \\ \overline{\beta} \\ y \end{pmatrix}.$$

It is clear that if $y < \alpha$ then $y \leq \beta$ since $y \equiv \pm r \pmod{m}$.

Eg: Let $k = 2$, $m = 10$ and $r = 3$

(i) $\begin{matrix} 10 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 7 \\ \overline{3} \\ 1 \end{matrix}$ (ii) $\begin{matrix} 33 \\ \overline{20} \\ 7 \end{matrix} \longrightarrow \begin{matrix} 33 \\ \overline{13} \\ \overline{7} \\ 7 \end{matrix}$

<u>Case 2:</u> If y is not overlined and $y \geq \alpha$, then we replace

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ by } \begin{pmatrix} y \\ x \end{pmatrix} \text{ and } \begin{pmatrix} \overline{x} \\ y \end{pmatrix} \text{ by } \begin{pmatrix} \overline{y} \\ x \end{pmatrix}.$$

Eg: Let $k = 2$, $m = 12$ and $r = 4$

(i) $\begin{matrix} 12 \\ 8 \end{matrix} \longrightarrow \begin{matrix} 8 \\ 12 \end{matrix} \longrightarrow \begin{matrix} 8 \\ \overline{8} \\ 4 \end{matrix}$ (ii) $\begin{matrix} \overline{24} \\ 20 \\ 4 \end{matrix} \longrightarrow \begin{matrix} \overline{20} \\ 24 \\ 4 \end{matrix} \longrightarrow \begin{matrix} \overline{20} \\ 16 \\ \overline{8} \\ 4 \end{matrix}$

<u>Case 3:</u> If y is overlined and $y < \beta$, then we replace

$$\begin{pmatrix} x \\ \overline{y} \end{pmatrix} \text{ by } \begin{pmatrix} \alpha \\ \overline{\beta} \\ \overline{y} \end{pmatrix} \text{ and } \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \text{ by } \begin{pmatrix} \overline{\alpha} \\ \overline{\beta} \\ \overline{y} \end{pmatrix}.$$

Eg: Let $k = 1$, $m = 8$ and $r = 3$

$$\text{(i)}\quad \begin{array}{c} 16 \\ \overline{3} \end{array} \longrightarrow \begin{array}{c} 11 \\ \overline{5} \\ \overline{3} \end{array} \qquad\qquad \text{(ii)}\quad \begin{array}{c} \overline{24} \\ \overline{5} \\ 3 \end{array} \longrightarrow \begin{array}{c} \overline{13} \\ \overline{11} \\ \overline{5} \\ 3 \end{array}$$

<u>Case 4:</u> If y is overlined and $y \geq \beta$, then we replace

$$\begin{pmatrix} x \\ \overline{y} \end{pmatrix} \text{ by } \begin{pmatrix} \dfrac{y+m}{x-m} \end{pmatrix} \text{ and } \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \text{ by } \begin{pmatrix} \dfrac{\overline{y+m}}{x-m} \end{pmatrix}.$$

Eg: Let $k = 1$, $m = 9$ and $r = 4$

$$\text{(i)}\quad \begin{array}{c} 18 \\ \overline{5} \end{array} \longrightarrow \begin{array}{c} 14 \\ \overline{9} \end{array} \longrightarrow \begin{array}{c} 14 \\ \overline{5} \\ \overline{4} \end{array} \qquad\qquad \text{(ii)}\quad \begin{array}{c} \overline{27} \\ \overline{14} \\ 4 \end{array} \longrightarrow \begin{array}{c} \overline{23} \\ \overline{18} \\ 4 \end{array} \longrightarrow \begin{array}{c} \overline{23} \\ \overline{13} \\ \overline{5} \\ 4 \end{array}$$

This process is continued till x splits into (α, β) where $\alpha, \beta \equiv \pm r \pmod{m}$.
Eg: Let $k = 3$, $m = 6$ and $r = 2$

$$\begin{array}{c} \overline{34} \\ 30 \\ \left.\begin{array}{c} 24 \\ 20 \end{array}\right\} \\ \overline{10} \end{array} \longrightarrow \begin{array}{c} \overline{34} \\ 30 \\ 20 \\ \left.\begin{array}{c} 24 \\ \overline{10} \end{array}\right\} \end{array} \longrightarrow \begin{array}{c} \overline{34} \\ 30 \\ 20 \\ 16 \\ \left.\overline{18}\right\} \end{array} \longrightarrow \begin{array}{c} \overline{34} \\ 30 \\ 20 \\ 16 \\ \overline{10} \\ \overline{8} \end{array}$$

Step DC_3. Look for the next multiple of m say x'. The same procedure explained in the Step DC_2 is carried out till x' splits into (α, β).

We apply Step DC_2 and Step DC_3 till all the parts $\equiv 0 \pmod{m}$ in ψ are split. The resulting partition will be a partition enumerated by $C_{k,r,m}(n)$.

From the above procedure it is clear that the map $D_{k,r,m} \to C_{k,r,m}$ is the inverse map of $C_{k,r,m} \to D_{k,r,m}$.

We now illustrate the reverse map by taking the same partition,

$\psi = \overline{160}+107+100+60+\overline{33}+\overline{13}+13$ of $D_{4,7,20}(486)$ where $k = 4, m = 20$ and $r = 7$ obtained from

$$\pi = \overline{107} + 87 + \overline{73} + 53 + 53 + \overline{47} + \overline{33} + \overline{13} + 13 + \overline{7}$$

$$D_{k,r,m}(n) \to C_{k,r,m}(n)$$

$$\begin{array}{c} \overline{160} \\ 107 \\ 100 \\ \left.\begin{array}{c} 60 \\ \overline{33} \end{array}\right\} \\ \overline{13} \\ 13 \end{array} \rightarrow \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \left.\begin{array}{c} \overline{40} \\ \overline{13} \end{array}\right\} \\ 13 \end{array} \rightarrow \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \overline{33} \\ \left.\begin{array}{c} \overline{20} \\ 13 \end{array}\right\} \end{array} \rightarrow \begin{array}{c} \overline{160} \\ 107 \\ 100 \\ 53 \\ \overline{33} \\ \overline{13} \\ \left.20\right\} \end{array} \rightarrow \begin{array}{c} \overline{160} \\ 107 \\ \left.\begin{array}{c} 100 \\ 53 \end{array}\right\} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \rightarrow \begin{array}{c} \overline{160} \\ 107 \\ 53 \\ \left.\begin{array}{c} 100 \\ \overline{33} \end{array}\right\} \\ \overline{13} \\ 13 \\ \overline{7} \end{array} \rightarrow$$

$$\begin{array}{c} \left.\begin{array}{c}\overline{160}\\ 107\end{array}\right\} \\ 53 \\ 53 \\ \overline{47} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{\overline{7}} \end{array} \quad \longrightarrow \quad \begin{array}{c} \overline{107} \\ \left.\begin{array}{c}160\\ 53\end{array}\right\} \\ 53 \\ \overline{47} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{\overline{7}} \end{array} \quad \longrightarrow \quad \begin{array}{c} \overline{107} \\ 87 \\ \overline{73} \\ 53 \\ 53 \\ \overline{47} \\ \overline{33} \\ \overline{13} \\ 13 \\ \overline{\overline{7}} \end{array}$$

The last partition is the associated partition of ψ enumerated by $C_{4,7,20}(486)$.

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